第40卷 第3期
2023年6月

应 用 力 学 学 报
Chinese Journal of Applied Mechanics

Vol. 40 No. 3
Jun. 2023

文章编号:1000-4939(2023)03-0663-08

基于非线性本构的单壁碳纳米管的Euler-Bernoulli梁模型

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摘 要: 在近期石墨烯的力学实验中,发现其在有限变形的条件下应力应变本构关系是非线性的。石墨烯卷曲形成碳纳米管,大管径单壁碳纳米管的应力应变关系,在理论上和石墨烯一致。在本研究中,基于石墨烯的非线性本构关系,建立了新的单壁碳纳米管的Euler-Bernoulli梁模型。之后对两端铰支的情况,使用Galerkin方法和多尺度法研究单壁碳纳米管在均布载荷作用下的静力弯曲和受迫振动问题。结果显示,在静力弯曲时,本构中的非线性项对碳纳米管起到了刚度软化的作用。在受迫振动时,本构中的非线性项改变了振幅分岔点的位置。因此本构中的非线性项对碳纳米管力学性质的显著影响不容忽视。

关键词: 单壁碳纳米管;非线性本构关系;Galerkin方法;多尺度法;Euler-Bernoulli梁

中图分类号: O322;O369 **文献标志码:** A **DOI:** 10.11776/j.issn.1000-4939.2023.03.019

Euler-Bernoulli beam model of single-walled carbon nanotubes based on the nonlinear constitutive model

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Abstract: Recent experiments have shown that the stress-strain relation of graphene is nonlinear under finite displacements. Theoretically, the stress-strain relation of single-walled carbon nanotubes (SWCNTs) with big tube diameters are consistent with that of graphene because the SWCNTs are formed by rolling graphene. In the present paper, we propose a new Euler-Bernoulli beam model of the SWCNTs based on the nonlinear stress-strain relation of graphene. Then, the static bending and forced vibrations of the SWCNTs are studied by the Galerkin method and the multi-scale method for the case of a hinged-hinged beam. The results indicate that the nonlinear terms of stress-strain relation can soften the stiffness of the SWCNTs during static bending. In the case of forced vibration, the nonlinear terms of stress-strain relation can change the position of the amplitude bifurcation points. Therefore, the nonlinear term in nonlinear constitu-

收稿日期: 2021-03-20 **修回日期:** 2021-04-24
基金项目: 国家自然科学基金资助项目(No. 11562009)
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引用格式: 王腾飞,黄坤,王明光,等. 基于非线性本构的单壁碳纳米管的Euler-Bernoulli梁模型[J]. 应用力学学报,2023,40(3):663-670.
WANG Tengfei, HUANG Kun, WANG Mingguang, et al. Euler-Bernoulli beam model of single – walled carbon nanotubes based on the nonlinear constitutive model[J]. Chinese journal of applied mechanics, 2023, 40(3):663-670.

tive cannot be ignored.

Key words: single-walled carbon nanotube; nonlinear stress-strain relation; galerkin method; multi-scale method; Euler-Bernoulli beam

在 1991 年,日本名城大学教授饭岛在做实验时,偶然发现了一种奇特结构的一维量子材料,首次把其命名为碳纳米管。自此以来,就因其众多优异的化学和力学等性能受到众多研究人员的关注[1-6],应用范围非常广泛,因此澄清其力学行为对于碳纳米管的应用具有重大意义。碳纳米管力学行为相比较传统材料比较复杂,目前针对它的研究方法比较有限,主要分为两种:理论分析和实验研究。对于实验研究,如 TREACY 等[7]使用 TEM 测量碳纳米管自由端热致振动的振幅,计算出碳管的弹性模量。WONG 等[8]在碳纳米管上不同位置处使用 AFM 探针施加不同作用力,得到了不同作用力下的碳纳米管对应挠度数值。然而,昂贵的实验设备和较高的实验失败率影响了此种方法的应用,即在纳米尺度上通过实验获得碳纳米管的力学性质是极其困难。因此,现在研究碳纳米管方法主要是理论分析,主要的方法有分子动力学法 、连续介质力学等。对于分子动力学方法,如 YAKOBSON 等[9]在研究碳纳米管轴压弯曲时,采用数值计算方法-分子动力学法得到了其能量曲线及复杂的屈曲位型。KOWAKI 等[10]对不同半径的碳纳米管在不同温度下进行了分子动力学模拟,并根据径向分布函数、均方位移和原子构型的温度依赖性估计了“熔化温度”。但这种方法计算过程复杂且耗时较长,对计算机运算能力要求极高,在成本和时间上受到很大的制约,因此连续介质模型在纳米结构分析中应用极其广泛。在连续介质力学中,离散的原子结构被假定为均匀和连续的[11]。黄坤等[12-13]在研究石墨烯时把其视为均匀连续体,并使用薄膜理论和板壳理论研究石墨烯的力学性质。黄坤等[14]在非局部微分本构关系的基础上,首次提出了具有小初始曲率的伯努利-欧拉纳米梁的非线性偏微分积分方程模型。然后应用该模型对单壁碳纳米管的静态弯曲和自由振动频率进行研究。余阳等[15]基于非局部应变梯度欧拉梁模型研究了充流单壁碳纳米管的自由振动和波动特性。ARROYO 等[16]基于 Cauchy-Born 准则研究碳纳米管发现连续介质模型与有限元方法相结合的模拟结果与涉及严重变形的零温度原子计算结果非常吻合。YAN 等[17-18]基于非局部弹性理论推导了双壁碳纳米管的动力学控制方程,并考虑了内外壁之间的范德华相互作用,研究了压力驱动定常流的失稳现象。

碳纳米管在几何上可以看成是二维的石墨烯按照不同方向卷曲而成。有研究表明,当碳纳米管的管径增加达到一定的数值时,碳纳米管众多的力学性能参数将与石墨片层对应的力学参数非常接近,比如弹性模量等,因此石墨烯与碳纳米管的应力应变关系理论上是一致的。在近期的石墨烯力学实验中,发现其在有限变形的条件下应力应变本构关系是非线性的[19-21]。对于单层石墨烯片,LEE 等[19]通过单轴实验拉伸得到其二次非线性本构关系,之后 CADELANO 等[20]通过比较原子模拟与连续介质弹性理论,得到了 LEE 本构方程中的所有非线性弹性系数。本研究将通过文献[19,21]中提出的包含三次非线性项的非线性本构关系,来建立单壁碳纳米管的非线性梁模型。然后应用该模型对单壁碳纳米管的静态弯曲和受迫振动进行了研究。

1 单壁碳纳米管的 Euler-Bernoulli 梁理论

本研究两端不可移动的单壁碳纳米管(如图 1 所示)来建模。l 设为碳纳米管的长度,$F(x,t)$ 为碳纳米管所受的分布载荷。

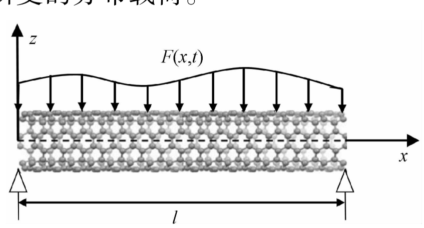

图 1 受均布荷载作用下的单壁碳纳米管模型

Fig. 1 Single-walled carbon nanotube model under uniform load

由文献[19,21],可得碳纳米管的非线性本构为

$$\sigma_{xx} = E\varepsilon_{xx} + D\varepsilon_{xx}^2 + G\varepsilon_{xx}^3 \tag{1}$$

其中:E 为二阶线弹性系数;D 为三阶弹性系数;G 为四阶弹性系数;σ_{xx} 和 ε_{xx} 分别为轴向应力和应变。

设碳纳米管 x、y、z 方向的位移分量分别为 u、v、w,这里的 v 远小于 u、w,因此本研究不考虑 v 的影响。根据 Euler-Bernoulli 梁理论,可把位移场写为[22]

$$u_1 = u(x,t) - z\frac{\partial w}{\partial x},\quad u_2 = 0,\quad u_3 = w(x,t) \tag{2}$$

其中：u_1、u_2 和 u_3 分别为 x、y、z 方向的位移；t 代表时间。在 Euler-Bernoull 梁理论中，横向剪切应变和横向法应变将被忽略。采用 von Kármán 位移应变关系[22-23]，x 方向的应变为

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \tag{3}$$

由式(1)可以得到有关轴向力和弯矩的表达式，分别用字母 N 和 M 表示，即

$$N = \iint_A \sigma_{xx}\mathrm{d}A = \iint_A E\varepsilon_{xx} + D\varepsilon_{xx}^2 + G\varepsilon_{xx}^3\mathrm{d}A \tag{4}$$

$$M = \iint_A z\sigma_{xx}\mathrm{d}A = \iint_A z(E\varepsilon_{xx} + D\varepsilon_{xx}^2 + G\varepsilon_{xx}^3)\mathrm{d}A \tag{5}$$

其中，A 是碳纳米管的横截面积。

将式(3)带入式(4)～(5)得到轴力和弯矩，为

$$N = EA\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + DA\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]^2 + GA\left[\left(\frac{\partial u}{\partial x}\right)^3 + \frac{1}{8}\left(\frac{\partial w}{\partial x}\right)^6 + \frac{3}{2}\left(\frac{\partial u}{\partial x}\right)^2\left(\frac{\partial w}{\partial x}\right)^2 + \frac{3}{4}\left(\frac{\partial w}{\partial x}\right)^4\frac{\partial u}{\partial x}\right] + I_2 D\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 3I_2 G\left(\frac{\partial^2 w}{\partial x^2}\right)^2\frac{\partial u}{\partial x} + \frac{3}{2}I_2 G\left(\frac{\partial^2 w}{\partial x^2}\right)^2\left(\frac{\partial w}{\partial x}\right)^2 \tag{6}$$

$$M = -EI_2\frac{\partial^2 w}{\partial x^2} - 2I_2 D\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]\frac{\partial^2 w}{\partial x^2} - 3I_2 G\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^2 w}{\partial x^2} - \frac{3}{4}I_2 G\left(\frac{\partial w}{\partial x}\right)^4\frac{\partial^2 w}{\partial x^2} - 3I_2 G\frac{\partial u}{\partial x}\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial w}{\partial x}\right)^2 - I_4 G\left(\frac{\partial^2 w}{\partial x^2}\right)^3 \tag{7}$$

其中：$I_2 = \iint_A z^2\mathrm{d}A$；$I_4 = \iint_A z^4\mathrm{d}A$。

对于 Euler-Bernoulli 梁，可以通过虚功原理得到运动方程[22,24]，为

$$\frac{\partial N}{\partial x} = m\frac{\partial^2 u}{\partial t^2} \tag{8}$$

$$\frac{\partial^2 M}{\partial x^2} + N\frac{\partial^2 w}{\partial x^2} + F(x,t) = m\frac{\partial^2 w}{\partial t^2} \tag{9}$$

其中：m 是碳纳米管的质量；$F(x,t)$ 是坐标轴的 z 方向的荷载，并忽略 x 方向上的载荷，本研究不考虑 y 方向上的载荷，如图 1 所示。把式(6)～(7)代入式(8)～(9)，得

$$\begin{aligned}\frac{\partial N}{\partial x} = {} & EA\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}\right] + DA\left[2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial w}{\partial x}\right)^3\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 u}{\partial x^2}\left(\frac{\partial w}{\partial x}\right)^2 + 2\frac{\partial u}{\partial x}\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}\right] + 2I_2 D\frac{\partial^2 w}{\partial x^2}\frac{\partial^3 w}{\partial x^3} + {} \\ & GA\left[3\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^2 u}{\partial x^2} + \frac{3}{4}\left(\frac{\partial w}{\partial x}\right)^5\frac{\partial^2 w}{\partial x^2} + 3\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}\left(\frac{\partial w}{\partial x}\right)^2 + 3\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2} + \frac{3}{4}\left(\frac{\partial w}{\partial x}\right)^4\frac{\partial^2 u}{\partial x^2} + 3\left(\frac{\partial w}{\partial x}\right)^3\frac{\partial^2 w}{\partial x^2}\frac{\partial u}{\partial x}\right] + {} \\ & 6I_2 G\frac{\partial^2 w}{\partial x^2}\frac{\partial^3 w}{\partial x^3}\frac{\partial u}{\partial x} + 3I_2 G\left(\frac{\partial^2 w}{\partial x^2}\right)^2\frac{\partial^2 u}{\partial x^2} + 3I_2 G\left(\frac{\partial^2 w}{\partial x^2}\right)^3\frac{\partial w}{\partial x} + 3I_2 G\frac{\partial^2 w}{\partial x^2}\frac{\partial^3 w}{\partial x^3}\left(\frac{\partial w}{\partial x}\right)^2 = m\frac{\partial^2 u}{\partial t^2}\end{aligned} \tag{10}$$

$$\begin{aligned}\frac{\partial^2 M}{\partial x^2} + N\frac{\partial^2 \omega}{\partial x^2} + F(x,t) = {} & -EI_2\frac{\partial^4 w}{\partial x^4} - 2I_2 D\left[\frac{\partial^3 u}{\partial x^3}\frac{\partial^2 w}{\partial x^2} + 2\frac{\partial^2 u}{\partial x^2}\frac{\partial^3 w}{\partial x^3} + \frac{\partial u}{\partial x}\frac{\partial^4 w}{\partial x^4} + \left(\frac{\partial^2 w}{\partial x^2}\right)^3 + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\frac{\partial^4 w}{\partial x^4} + {}\right. \\ & \left. 3\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}\frac{\partial^3 w}{\partial x^3}\right] - 3I_4 G\left[2\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial^3 w}{\partial x^3}\right)^2 + \left(\frac{\partial^2 w}{\partial x^2}\right)^2\frac{\partial^4 w}{\partial x^4}\right] - 3I_2 G\left[2\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^2 w}{\partial x^2} + 2\frac{\partial u}{\partial x}\frac{\partial^3 u}{\partial x^3}\frac{\partial^2 w}{\partial x^2} + 4\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}\frac{\partial^3 w}{\partial x^3} + {}\right. \\ & \left.\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^4 w}{\partial x^4} + 3\left(\frac{\partial w}{\partial x}\right)^2\left(\frac{\partial^2 w}{\partial x^2}\right)^3 + 3\left(\frac{\partial w}{\partial x}\right)^3\frac{\partial^2 w}{\partial x^2}\frac{\partial^3 w}{\partial x^3}\right] - 3I_2 G\left[\frac{1}{4}\left(\frac{\partial w}{\partial x}\right)^4\frac{\partial^4 w}{\partial x^4} + \frac{\partial^3 u}{\partial x^3}\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial w}{\partial x}\right)^2 + {}\right. \\ & \left. 2\frac{\partial^2 u}{\partial x^2}\frac{\partial^3 w}{\partial x^3}\left(\frac{\partial w}{\partial x}\right)^2 + 4\frac{\partial^2 u}{\partial x^2}\frac{\partial w}{\partial x}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \frac{\partial u}{\partial x}\frac{\partial^4 w}{\partial x^4}\left(\frac{\partial w}{\partial x}\right)^2 + 6\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}\frac{\partial^3 w}{\partial x^3}\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial x}\left(\frac{\partial^2 w}{\partial x^2}\right)^3\right] + {} \\ & \frac{\partial^2 w}{\partial x^2}\left\{EA\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + DA\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]^2 + GA\left[\left(\frac{\partial u}{\partial x}\right)^3 + \frac{1}{8}\left(\frac{\partial w}{\partial x}\right)^6 + \frac{3}{2}\left(\frac{\partial u}{\partial x}\right)^2\left(\frac{\partial w}{\partial x}\right)^2 + {}\right.\right. \\ & \left.\left.\frac{3}{4}\left(\frac{\partial w}{\partial x}\right)^4\frac{\partial u}{\partial x}\right] + I_2 D\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 3I_2 G\left(\frac{\partial^2 w}{\partial x^2}\right)^2\frac{\partial u}{\partial x} + \frac{3}{2}I_2 G\left(\frac{\partial^2 w}{\partial x^2}\right)^2\left(\frac{\partial w}{\partial x}\right)^2\right\} + F(x,t) = m\frac{\partial^2 w}{\partial t^2}\end{aligned} \tag{11}$$

为了简化讨论，式(10)中舍去所有的高次非线性项，得

$$EA\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}\right] + 2I_2 D\frac{\partial^2 w}{\partial x^2}\frac{\partial^3 w}{\partial x^3} = m\frac{\partial^2 u}{\partial t^2} \tag{12}$$

对于经典细长梁，纵向位移 u 主要由横向位移引起形变[24]，而且纵向惯性项可以忽略。此时，从式(12)可知

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2} - 2\lambda\frac{\partial^2 w}{\partial x^2}\frac{\partial^3 w}{\partial x^3} \tag{13}$$

对式(13)求一次导,得

$$\frac{\partial^3 u}{\partial x^3} = -\left(\frac{\partial^2 w}{\partial x^2}\right)^2 - \frac{\partial w}{\partial x}\frac{\partial^3 w}{\partial x^3} - 2\lambda\left(\frac{\partial^3 w}{\partial x^3}\right)^2 - 2\lambda\frac{\partial^2 w}{\partial x^2}\frac{\partial^4 w}{\partial x^4} \quad (14)$$

其中,令 $\lambda = \frac{ID}{EA}$。对式(13)积分两次,分别得

$$\frac{\partial u}{\partial x} = -\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - \lambda\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + c_1(t) \quad (15)$$

$$u = -\frac{1}{2}\int_0^x\left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x - \lambda\int_0^x\left(\frac{\partial^2 w}{\partial x^2}\right)^2 \mathrm{d}x + c_1(t)x + c_2(t) \quad (16)$$

其中,$c_1(t)$ 和 $c_2(t)$ 是时间的函数。对于两端铰支的梁,它的边界条件为

$$u(0,t) = u(l,t) = 0 \quad (17)$$

$$w(0,t) = w(l,t) = \frac{\partial^2 w(0,t)}{\partial x^2} = \frac{\partial^2 w(l,t)}{\partial x^2} = 0 \quad (18)$$

将边界条件式(17)~(18)代入式(16),可得

$$c_2(t) = 0 \quad (19)$$

$$c_1(t) = \frac{1}{2l}\int_0^l\left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x + \frac{\lambda}{l}\int_0^l\left(\frac{\partial^2 w}{\partial x^2}\right)^2 \mathrm{d}x \quad (20)$$

将式(11)中高次项全部舍去,得

$$-EI_2\frac{\partial^4 w}{\partial x^4} - 2I_2D\left[\frac{\partial^3 u}{\partial x^3}\frac{\partial^2 w}{\partial x^2} + 2\frac{\partial^2 u}{\partial x^2}\frac{\partial^3 w}{\partial x^3} + \frac{\partial u}{\partial x}\frac{\partial^4 w}{\partial x^4} + \left(\frac{\partial^2 w}{\partial x^2}\right)^3 + 3\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}\frac{\partial^3 w}{\partial x^3} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\frac{\partial^4 w}{\partial x^4}\right] + \frac{\partial^2 w}{\partial x^2}\left[EA\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + I_2D\left(\frac{\partial^2 w}{\partial x^2}\right)^2\right] - 3I_4G\left[2\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial^3 w}{\partial x^3}\right)^2 + \left(\frac{\partial^2 w}{\partial x^2}\right)^2\frac{\partial^4 w}{\partial x^4}\right] + F(x,t) = m\frac{\partial^2 w}{\partial t^2} \quad (21)$$

把式(13)~(15)带入式(21)得

$$m\frac{\partial^2 w}{\partial t^2} + 3I_4G\left[2\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial^3 w}{\partial x^3}\right)^2 + \left(\frac{\partial^2 w}{\partial x^2}\right)^2\frac{\partial^4 w}{\partial x^4}\right] - 12\lambda I_2D\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial^3 w}{\partial x^3}\right)^2 - 6\lambda I_2D\left(\frac{\partial^2 w}{\partial x^2}\right)^2\frac{\partial^4 w}{\partial x^4} + EI_2\frac{\partial^4 w}{\partial x^4} = \left[EA\frac{\partial^2 w}{\partial x^2} - 2I_2D\frac{\partial^4 w}{\partial x^4}\right]c_1 + F(x,t) \quad (22)$$

把式(20)带入式(22)得

$$m\frac{\partial^2 w}{\partial t^2} - 12\lambda I_2D\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial^3 w}{\partial x^3}\right)^2 - 6\lambda I_2D\left(\frac{\partial^2 w}{\partial x^2}\right)^2\frac{\partial^4 w}{\partial x^4} + 3I_4G\left[2\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial^3 w}{\partial x^3}\right)^2 + \left(\frac{\partial^2 w}{\partial x^2}\right)^2\frac{\partial^4 w}{\partial x^4}\right] + EI_2\frac{\partial^4 w}{\partial x^4} - EA\frac{\partial^2 w}{\partial x^2}\left[\frac{1}{2l}\int_0^l\left(\frac{\partial w}{\partial x}\right)^2\mathrm{d}x + \frac{\lambda}{l}\int_0^l\left(\frac{\partial^2 w}{\partial x^2}\right)^2\mathrm{d}x\right] + 2I_2D\frac{\partial^4 w}{\partial x^4}\left[\frac{1}{2l}\int_0^l\left(\frac{\partial w}{\partial x}\right)^2\mathrm{d}x + \frac{\lambda}{l}\int_0^l\left(\frac{\partial^2 w}{\partial x^2}\right)^2\mathrm{d}x\right] = F(x,t) \quad (23)$$

为了方便下面的讨论,对式(23)进行无量纲化处理,令 $\bar{x} = x/l$, $\bar{w} = w/l$, $\bar{t} = \omega_0 t$,其中,$\omega_0 = \sqrt{\frac{\pi^4 EI_2}{l^4 m}}$,这里的 ω_0 表示两端绞支梁的固有频率。式(23)化为

$$\frac{\partial^2 \bar{w}}{\partial \bar{t}^2} - \frac{12\lambda I_2D}{m\omega_0^2 l^6}\frac{\partial^2 \bar{w}}{\partial \bar{x}^2}\left(\frac{\partial^3 \bar{w}}{\partial \bar{x}^3}\right)^2 - \frac{6\lambda I_2D}{m\omega_0^2 l^6}\left(\frac{\partial^2 \bar{w}}{\partial \bar{x}^2}\right)^2\frac{\partial^4 \bar{w}}{\partial \bar{x}^4} + \frac{3I_4G}{m\omega_0^2 l^6}\left[2\frac{\partial^2 \bar{w}}{\partial \bar{x}^2}\left(\frac{\partial^3 \bar{w}}{\partial \bar{x}^3}\right)^2 + \left(\frac{\partial^2 \bar{w}}{\partial \bar{x}^2}\right)^2\frac{\partial^4 \bar{w}}{\partial \bar{x}^4}\right] + \frac{EI_2}{m\omega_0^2 l^4}\frac{\partial^4 \bar{w}}{\partial \bar{x}^4} - \frac{EA}{m\omega_0^2 l}\frac{\partial^2 \bar{w}}{\partial \bar{x}^2}\left[\frac{1}{2l}\int_0^1\left(\frac{\partial \bar{w}}{\partial \bar{x}}\right)^2\mathrm{d}\bar{x} + \frac{\lambda}{l^3}\int_0^1\left(\frac{\partial^2 \bar{w}}{\partial \bar{x}^2}\right)^2\mathrm{d}\bar{x}\right] + \frac{2I_2D}{m\omega_0^2 l^3}\frac{\partial^4 \bar{w}}{\partial \bar{x}^4}\left[\frac{1}{2l}\int_0^1\left(\frac{\partial \bar{w}}{\partial \bar{x}}\right)^2\mathrm{d}\bar{x} + \frac{\lambda}{l^3}\int_0^1\left(\frac{\partial^2 \bar{w}}{\partial \bar{x}^2}\right)^2\mathrm{d}\bar{x}\right] = \frac{1}{m\omega_0^2 l}F_1(\bar{x},\bar{t}) \quad (24)$$

由上可知,该梁模型边界条件为

$$\bar{w}(0,\bar{t}) = \bar{w}(1,\bar{t}) = \frac{\partial^2 \bar{w}(0,\bar{t})}{\partial \bar{x}^2} = \frac{\partial^2 w(1,\bar{t})}{\partial \bar{x}^2} = 0$$

在式(24)中是含有积分项的非线性偏微分方程,难以得到精确的解析解。在此,使用 Galerkin 法来近似求解。在归一化边界条件下,可设微分方程式(24)的近似解[23]为

$$\bar{w} = \sum_{n=1}^{\infty}\eta_n \sin n\pi\bar{x} \quad (25)$$

这里只取第一项,将式(25)带入式(24)中,再在方程的两边同时乘 $\sin\pi\bar{x}$,然后在区间[0,1]上积分(Galerkin 一次截断),可得到描述单壁碳纳米管一阶模态受迫振动的微分方程,为

$$\frac{\partial^2 \eta}{\partial \bar{t}^2} + \frac{EI_2\pi^4}{m\omega_0^2 l^4}\eta + \left[\frac{EA\pi^4}{4m\omega_0^2 l^2}\left(1 + \frac{4\lambda\pi^2}{l^2}\right) + \frac{3I_4G\pi^8}{4m\omega_0^2 l^6} - \frac{\lambda I_2D\pi^8}{2m\omega_0^2 l^6}\right]\eta^3 = \frac{2}{m\omega_0^2 l}\int_0^1 F_1(\bar{x},\bar{t})\sin\pi\bar{x}\,\mathrm{d}\bar{x} \quad (26)$$

上式可写为

$$\ddot{\eta} + k_1^2\eta + k_3\eta^3 = F \quad (27)$$

其中

$$k_3 = \frac{EA\pi^4}{4m\omega_0{}^2 l^2}\left(1 + \frac{4\lambda\pi^2}{l^2}\right) + \frac{3I_4 G\pi^8}{4m\omega_0{}^2 l^6} - \frac{\lambda I_2 D\pi^8}{2m\omega_0{}^2 l^6};$$

$$k_1^2 = \frac{EI_2\pi^4}{m\omega_0{}^2 l^4}; F = \frac{2}{m\omega_0{}^2 l}\int_0^1 F_1(\bar{x},\bar{t})\sin\pi\bar{x}\mathrm{d}\bar{x}$$

2　静力学分析

去掉式(27)的惯性项，可得碳纳米管的静力学方程

$$k_1^2\eta + k_3\eta^3 = F \tag{28}$$

在此为方便讨论，本研究以(15,15)碳纳米管为例，$d=2.034$ nm 为其直径的大小，$t_v=0.34$ nm 为其管壁的厚度。它的物理参数为：$E = 1$ TPa；$G = 6.58$ TPa[21]。其它参数为：$l = 11$ nm；$m = 4.866 \times 10^{-15}$ kg/m；$I_2 = 1.154$ nm^4；$I_4 = 0.953$ nm^6；$A = 2.171$ nm^2；$\omega_0^2 = 1.5778 \times 10^{24}$。

将这些参数代入得到

$$k_1 = 1, k_3 = 56.8826, D = G = 0$$

$$k_1 = 1, k_3 = 12.3536, D \neq 0, G = 0$$

$$k_1 = 1, k_3 = 15.6344, D \neq 0, G \neq 0$$

通过式(28)和上述的数值，即可得到碳纳米管的静力弯曲振幅，如下图 2 所示。

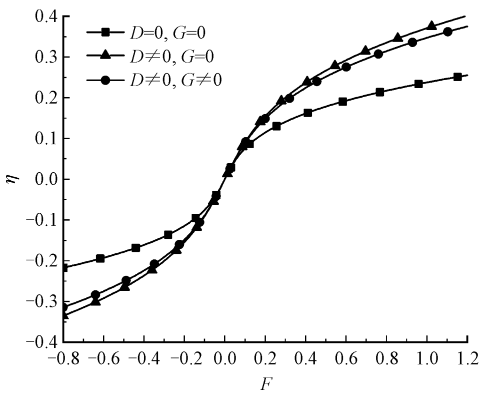

图 2　碳纳米管静力弯曲变形

Fig. 2　Bending deformations under static loads

由图 2 可知，碳纳米管应力应变非线性本构中的二次和三次非线性项均对碳纳米管的静力学性质具有非常明显的影响，这种影响程度会随外激励的增加而加深。

3　弯曲振动分析

在式(26)加阻尼项，可得碳纳米管在一阶模态下的有阻尼受迫振动方程，即

$$\frac{\partial^2\eta}{\partial\bar{t}^2} + C\frac{\partial\eta}{\partial\bar{t}} + \frac{EI_2\pi^4}{m\omega_0{}^2 l^4}\eta + \left[\frac{EA\pi^4}{4m\omega_0{}^2 l^2}\left(1 + \frac{4\lambda\pi^2}{l^2}\right) + \frac{3I_4 G\pi^8}{4m\omega_0{}^2 l^6} - \frac{\lambda I_2 D\pi^8}{2m\omega_0{}^2 l^6}\right]\eta^3 = \frac{2}{m\omega_0^2 l}\int_0^1 F_1(\bar{x},\bar{t})\sin\pi\bar{x}\mathrm{d}\bar{x} \tag{29}$$

其中，C 为阻尼常数。

为了便于摄动求解方程，令 $C = 2\varepsilon^2 c$，$F_1(\bar{x},\bar{t}) = F_2(\bar{x})\cos\omega\bar{t}$。因此上式方程可写为

$$\frac{\partial^2\eta}{\partial\bar{t}^2} + 2\varepsilon^2 c\frac{\partial\eta}{\partial\bar{t}} + k_1^2\eta + k_3\eta^3 = \varepsilon^3 f\cos\omega\bar{t} \tag{30}$$

其中

$$k_3 = \frac{EA\pi^4}{4m\omega_0^2 l^2}\left(1 + \frac{4\lambda\pi^2}{l^2}\right) + \frac{3I_4 G\pi^8}{4m\omega_0^2 l^6} - \frac{\lambda I_2 D\pi^8}{2m\omega_0^2 l^6};$$

$$k_1^2 = \frac{EI_2\pi^4}{m\omega_0^2 l^4};\ \varepsilon^3 f = \frac{2}{m\omega_0^2 l}\int_0^1 F_2(\bar{x})\sin\pi\bar{x}\mathrm{d}\bar{x};$$

其中：ε 为摄动的小参数；c 为阻尼常数。在此使用多尺度方法求解式(30)，引入频差 σ，令 $\varepsilon^2\sigma = \omega - k_1$，把方程的解在两个时间尺度上展开为

$$\eta = \varepsilon\eta_0(T_0,T_2) + \varepsilon^3\eta_1(T_0,T_2) \tag{31}$$

其中，$T_n = \varepsilon^n\bar{t}$，$(n = 0,1,2,\cdots)$。

把式(31)代入式(30)中，令 ε^1，ε^3 的系数相等得到[23]

$$\varepsilon^1:\quad D_0^2\eta_0 + k_1^2\eta_0 = 0 \tag{32a}$$

$$\varepsilon^3:\quad D_0^2\eta_1 + k_1^2\eta_1(t) = -2D_0D_2\eta_0 - 2cD_0\eta_0 - k_3\eta_0^3 + f\cos(k_1T_0 + \sigma T_2) \tag{32b}$$

其中，D_0，D_2 分别表示对 T_0，T_2 求导。式(32a)的解可表示为

$$\eta_0 = A(T_2)\exp(\mathrm{i}k_1T_0) + cc \tag{33}$$

其中，cc 表示其左边各项的共轭复数。

把 η_0 代入式(32b)，将 $\cos(k_1T_0 + \sigma T_2)$ 以指数函数形式表示，得到

$$D_0^2\eta_1 + k_1^2\eta_1 = -[2\mathrm{i}k_1(D_2A + cA) + 3k_3A^2\bar{A}]\cdot\exp(\mathrm{i}k_1T_0) - k_3A^3\exp(3\mathrm{i}k_1T_0) + \frac{1}{2}f\exp[\mathrm{i}(k_1T_0 + \sigma T_2)] + cc \tag{34}$$

其中，cc 表示其左边各项的共轭复数。

为消除式(34)中的永年项，要求函数 A 满足

$$2\mathrm{i}k_1(D_2A + cA) + 3k_3A^2\bar{A} - \frac{1}{2}f\exp(\mathrm{i}\sigma T_2) = 0 \tag{35}$$

将复函数 A 表示为指数形式

$$A = \frac{1}{2}\alpha\exp(\mathrm{i}\beta) \tag{36}$$

其中,α 和 β 是待定的关于 T_2 的实函数。把所得结果的实部和虚部分开,得到 α 和 β 一阶常微分方程组

$$D_2\alpha = -c\alpha + \frac{1}{2}\frac{f}{k_1}\sin(\sigma T_2 - \beta) \tag{37a}$$

$$\alpha D_2\beta = \frac{3}{8}\frac{k_3}{k_1}\alpha^3 - \frac{1}{2}\frac{f}{k_1}\cos(\sigma T_2 - \beta) \tag{37b}$$

将式(36)代入式(33),则解式(31)得第一次近似解

$$\eta = \varepsilon\alpha\cos(k_1 t + \beta) + O(\varepsilon^2) \tag{38}$$

其中,$O(\ )$表示为同阶无穷小。

令 $\gamma = \sigma T_2 - \beta$ 得,

$$D_2\alpha = -c\alpha + \frac{1}{2}\frac{f}{k_1}\sin\gamma \tag{39a}$$

$$\alpha D_2\gamma = \sigma\alpha + \frac{3}{8}\frac{k_3}{k_1}\alpha^3 - \frac{1}{2}\frac{f}{k_1}\cos\gamma \tag{39b}$$

令 $D_2\alpha = D_2\gamma = 0$,系统的稳态解为

$$c\alpha = \frac{1}{2}\frac{f}{k_1}\sin\gamma \tag{40a}$$

$$\frac{3}{8}\frac{k_3}{k_1}\alpha^3 - \alpha\sigma = -\frac{1}{2}\frac{f}{k_1}\cos\gamma \tag{40b}$$

上面二式的平方后求和得

$$\left[c^2 + \left(\sigma - \frac{3}{8}\frac{k_3}{k_1}\alpha^2\right)^2\right]\alpha^2 = \frac{f^2}{4k_1^2} \tag{41}$$

设 $\alpha \neq 0$ 时,把式(41)整理为

$$\sigma = \frac{3}{8}\frac{k_3}{k_1}\alpha^2 \pm \left(\frac{f^2}{4k_1^2\alpha^2} - c^2\right)^{\frac{1}{2}} \tag{42}$$

这里仍以(15,15)单壁碳纳米管为例,基本参数与静力分析所举实例一致。对于碳纳米管的阻尼系数,为了简化讨论,在此取 $C = 0.05$,$\varepsilon = 0.1$,$f = 5$得到对应的频响曲线如图 3 所示。

由图 3 可知,当 σ 处于某一临界值时,振幅将出现分岔,而本构中的非线性项改变了振幅分岔点的位置。在分岔点处,碳纳米管的振幅会伴随参数的轻微变化而突然改变,这对结构振动产生了显著影响。为考查摄动解的准确性,对式(29)进行数值计算。在图 3 上选取点 $\sigma = 3.76$,对 3 种不同参数进行数值计算。结果显示,摄动解准确地描述碳管的受迫振动行为,如图 4 所示。此外,图 3 还显示,本构中的非线性项起到刚度软化的作用。

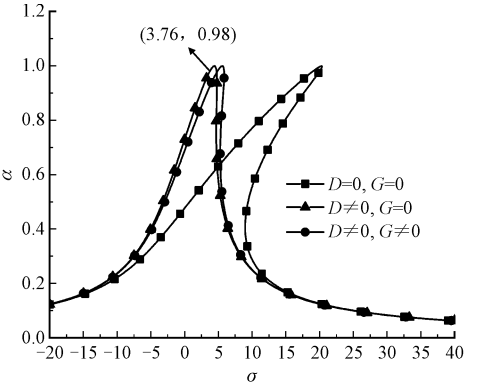

图 3 外激励 $f = 5$ 时的频响曲线

Fig. 3 Frequency response curve for $f = 5$

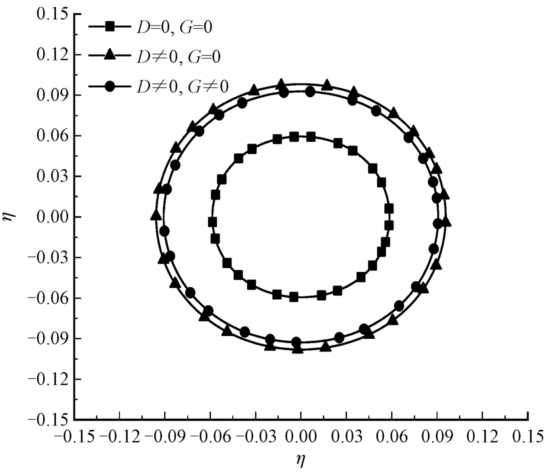

图 4 相位图在 $\sigma = 3.76$

Fig. 4 Phase diagram for $\sigma = 3.76$

通过式(41),并取定 $\sigma = 5$ 的数值,可得到对应的激励振幅响应曲线,如图 5 所示。

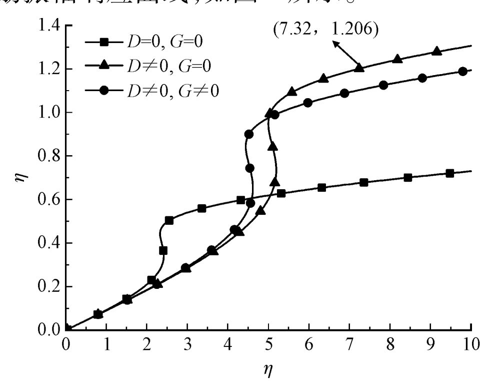

图 5 当 $\sigma = 5$ 时的外激励响应曲线

Fig. 5 Response curve of external excitation for $\sigma = 5$

该图显示,对于相同的 σ 值,随着外激励值的增加,响应曲线都出现了分岔点,但分岔点的位置并不同。这说明本构中的非线性项同样改变了这种情况下分岔点的位置。为检验解析解的准确性,在图 5 上选取$f = 7.32$ 对不同的参数对式(29)进行数值计算,结果同样显示,解析解具有良好的精度,如图 6

所示。

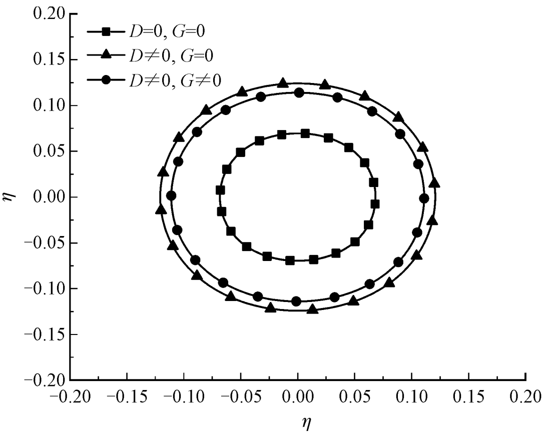

图 6　当 σ = 5 时的相位图

Fig. 6　Phase diagram for σ = 5

4 结　论

在本研究中，我们通过石墨烯的非线性应力应变关系和经典 Euler-Bernoulli 梁的平衡方程，建立了新的单壁碳纳米管的非线性梁模型。之后通过 Galerkin 方法和多尺度法研究了两端铰支条件下，(15,15) 单壁碳纳米管在垂直于轴线的分布力作用下的静力弯曲和一阶模态受迫振动问题。本研究得到 3 点主要结论。

1) 本构中的非线性项对碳纳米管的静力学有显著的影响，起到了刚度软化的作用，随着外载荷的增加而愈加明显。

2) 在受迫振动时，本构中的非线性项改变了振幅分岔点的位置，这对结构振动将产生显著影响。

3) 本研究结果表明，对基于线性本构关系的单壁碳纳米管研究，有必要重新检视其准确性。

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（编辑 史淑英）